

## Homework 8

### 1. *Recursive estimation for navigation*

We have a ship at position

$$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

and beacons located at

$$q_1 = \begin{bmatrix} 500 \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} 0 \\ 500 \end{bmatrix} \quad q_3 = \begin{bmatrix} 500 \\ -200 \end{bmatrix} \quad q_4 = \begin{bmatrix} -500 \\ 500 \end{bmatrix}$$

We take measurements at times  $t = 1, \dots, 6$ . At each time  $t$ ,  $r(t)$  is a vector of range measurements, and  $b(t)$  is a vector indicating which beacons were used. e.g.

$$r(1) = \begin{bmatrix} 498.373 \\ 497.219 \\ 498.116 \end{bmatrix} \quad b(1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

indicates that at time 1, the distance to beacon 1 was measured at 498.373, the distance to beacon 2 was measured at 497.18 and the distance to beacon 1 was measured again at 498.116.

Measurements to beacon  $q_i$  are corrupted by Gaussian noise of covariance  $c_i$ , where

$$c_1 = 0.1 \quad c_2 = 0.15 \quad c_3 = 0.2 \quad c_4 = 0.1$$

Each scalar measurement is corrupted by noise independent from that affecting any other scalar measurement. So in the above example, we would have a covariance matrix for the measurement of

$$\Sigma = \begin{bmatrix} 0.1 & & \\ & 0.15 & \\ & & 0.1 \end{bmatrix}$$

The data is

$$b(1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad b(2) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad b(3) = 2 \quad b(4) = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \quad b(5) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad b(6) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and

$$r(1) = \begin{bmatrix} 498.3732 \\ 497.2192 \\ 498.1156 \end{bmatrix} \quad r(2) = \begin{bmatrix} 498.4453 \\ 537.5662 \end{bmatrix} \quad r(3) = 496.9780$$

$$r(4) = \begin{bmatrix} 536.9246 \\ 537.6320 \\ 537.7702 \end{bmatrix} \quad r(5) = \begin{bmatrix} 706.2695 \\ 496.9436 \end{bmatrix} \quad r(6) = \begin{bmatrix} 497.5740 \\ 496.8541 \\ 538.5130 \end{bmatrix}$$

Denote by  $\hat{x}(t)$  the minimum-variance unbiased estimate of  $x$  given measurements  $y(1), \dots, y(t)$ , and by  $Q(t)$  the measurement covariance.

- Find the MMSE estimate  $\hat{x}(1)$  and  $Q(1)$  assuming the prior covariance is infinitely large.
- Find, using recursive estimation,  $\hat{x}(2), \dots, \hat{x}(6)$ , and  $Q(2), \dots, Q(6)$ . The 90% confidence ellipsoid at time  $t$  is

$$E(t) = \left\{ z \in \mathbb{R}^2 \mid (z - \hat{x}(t))^T Q(t)^{-1} (z - \hat{x}(t)) \leq F_{\chi^2}^{-1}(0.9) \right\}$$

For each  $t$  plot  $\hat{x}(t)$  and the 90% confidence ellipsoid.

- (c) Form the vector  $y$  and matrix  $A$  such that

$$y = Ax + w$$

where  $y$  contains all measurements. Compute the (non-recursive) MMSE estimate for  $x$ , and verify that it equals  $\hat{x}(6)$  that you computed recursively above.

- (d) Recall that an  $n \times n$  matrix  $J$  is called a *permutation* matrix if every row and every column contains exactly one 1, with every other entry 0. If we have

$$y_{\text{perm}} = Jy$$

Then  $y_{\text{perm}}$  just contains the entries of  $y$  in a different order. Use this to show that if we receive measurements in a different order, the final estimate  $\hat{x}(6)$  is unchanged.

- (e) Is it true that  $E(t) \subset E(t+1)$ ? That is, when we receive a new measurement, will the new confidence ellipsoid be inside the old one? Give a proof, or a counterexample.

## 2. *Downdating of estimates*

We have seen that one can use recursive estimation to update an estimate using new measurements. In this question, we look at what happens if you find out that one of your old measurements was incorrect. The effect of the incorrect measurement can be removed by *downdating*.

Suppose we have

$$y = Ax + w \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where  $w \sim \mathcal{N}(0, I)$ ,  $x \sim \mathcal{N}(0, \Sigma_x)$ ,  $y_1 \in \mathbb{R}^{m_1}$  and  $y_2 \in \mathbb{R}^{m_2}$ . We form  $\hat{x}_{12}$ , the MMSE estimate of  $x$  given measurements  $y_1$  and  $y_2$  using the standard MMSE

$$\begin{aligned} \hat{x}_{12} &= Q_{12} A^T y \\ Q_{12} &= (A^T A + \Sigma_x^{-1})^{-1} \end{aligned}$$

After doing this, we realize that  $y_1$  was really an erroneous measurement, and we'd prefer to use the estimate  $\hat{x}_2$  based on  $y_2$  only, given by

$$\begin{aligned} \hat{x}_2 &= Q_2 A_2^T y_2 \\ Q_2 &= (A_2^T A_2 + \Sigma_x^{-1})^{-1} \end{aligned}$$

Show that one may find  $Q_2$  and  $\hat{x}_2$  using

$$\begin{aligned} Q_2 &= Q_{12} + Q_{12} A_1^T (I - A_1 Q_{12} A_1^T)^{-1} A_1 Q_{12} \\ \hat{x}_2 &= \hat{x}_{12} - Q_{12} A_1^T (I - A_1 Q_{12} A_1^T)^{-1} (y_1 - A_1 \hat{x}_{12}) \end{aligned}$$

This removes the effect of  $y_1$ , and gives the optimal estimate just based on  $y_2$ . Notice that one only needs the measurement  $y_1$  that is being removed, not the other measurement  $y_2$ .