14 - Gaussian Stochastic Processes

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Linear systems driven by IID noise

consider the linear dynamical system

$$x(t+1) = Ax(t) + Bu(t) + v(t)$$

with

- the input $u(0), u(1), \ldots$ is not random
- the disturbance $v(0), v(1), v(2), \ldots$ is white Gaussian noise

$$\mathbf{E}\, v(t) = \mu_v(t) \qquad \mathbf{cov}\, v(t) = \Sigma_v$$

- the initial state is random $x(0) \sim \mathcal{N}(\mu_x(0), \Sigma_x(0))$, independent of $v(t)$ for all t

view this as *stochastic* simulation of the system

- what are the statistical properties (mean and covariance) of $x(t)$?

Evolution of Mean and Covariance

we have

$$x(t+1) = Ax(t) + Bu(t) + v(t)$$

taking the expectation of both sides, we have, as before

$$\boxed{\mu_x(t+1) = A\mu_x(t) + Bu(t) + \mu_v(t)}$$

taking the covariance of both sides, we have

$$\boxed{\Sigma_x(t+1) = A\Sigma_x(t)A^T + \Sigma_v}$$

i.e, the state covariance $\Sigma_x(t) = \mathbf{cov}(x(t))$ obeys a *Lyapunov recursion*

State Covariance

The solution to the Lyapunov recursion is

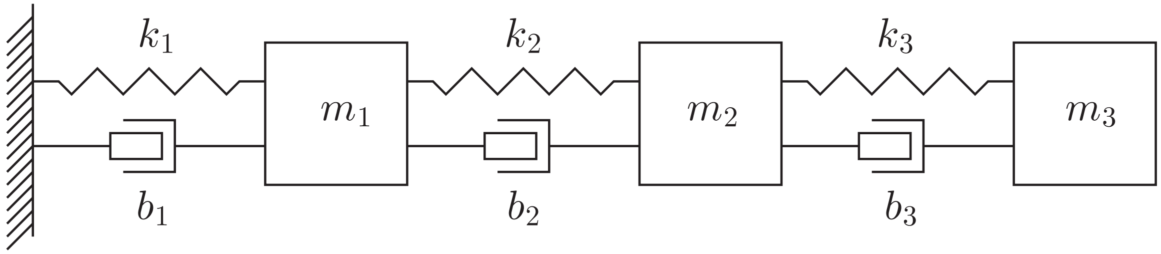
$$\Sigma_x(t) = A^t \Sigma_x(0)(A^t)^T + \sum_{k=0}^{t-1} A^k \Sigma_v (A^k)^T$$

Because the covariance of the state $\Sigma_x(t) = \mathbf{cov}\, x(t)$ is

$$\Sigma_x(t) = A^t \Sigma_x(0)(A^t)^T + \begin{bmatrix} A^{t-1} & \ldots & A & I \end{bmatrix} \begin{bmatrix} \Sigma_v & & & \\ & \Sigma_v & & \\ & & \ddots & \\ & & & \Sigma_v \end{bmatrix} \begin{bmatrix} (A^{t-1})^T \\ \vdots \\ A^T \\ I \end{bmatrix}$$

$$= A^t \Sigma_x(0)(A^t)^T + \sum_{k=0}^{t-1} A^k \Sigma_v (A^k)^T$$

Example: Mass-Spring System



masses $m_i = 1$, springs $k_i = 2$, dampers $b_i = 3$

$$\dot{x}(t) = A_c x(t) + B_{c1} w(t) + B_{c2} u(t)$$

where

$$A_c = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 2 & 0 & -6 & 3 & 0 \\ 2 & -4 & 2 & 3 & -6 & 3 \\ 0 & 2 & -2 & 0 & 3 & -3 \end{bmatrix} \qquad B_{c1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B_{c2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$u(t)$ is deterministic force applied to mass 3

$w(t) \in \mathbb{R}^3$ is random forcing $w(t) \sim \mathcal{N}(0, 0.2I)$ applied to all masses

Example: Mass-Spring System

discretization

$$x(t+1) = Ax(t) + B_1 w(t) + B_2 u(t)$$

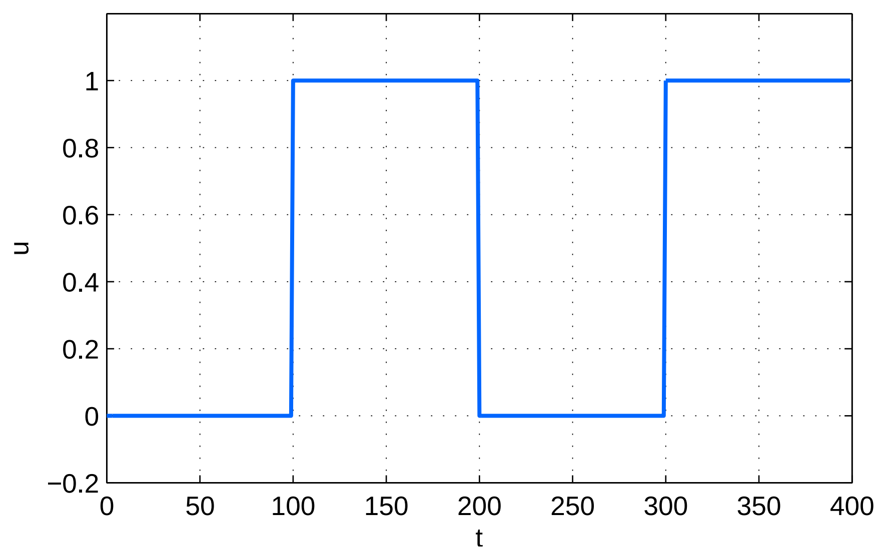
let $v(t) = B_1 w(t)$, so

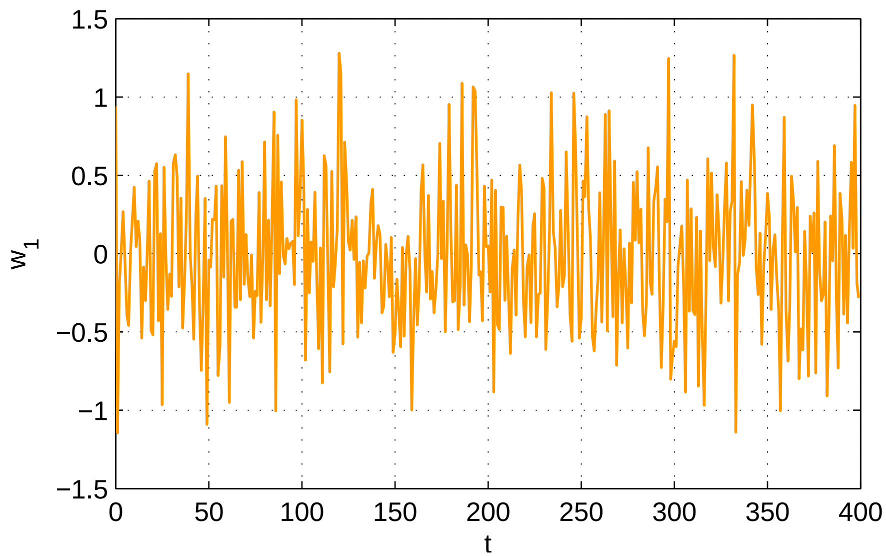
$$\mathbf{E}\, v(t) = 0 \qquad \mathbf{cov}\, v(t) = B_1 \Sigma_w B_1^T$$

and we have

$$x(t+1) = Ax(t) + B_2 u(t) + v(t)$$

the inputs are





Example: Mass-Spring System

simulate three things

- the evolution of the mean $\mu_x(t+1) = A\mu_x(t) + Bu(t) + \mu_v(t)$
- the evolution of the covariance $\Sigma_x(t+1) = A\Sigma_x(t)A^T + \Sigma_v$
- the state trajectory for a particular *realization* of the random process

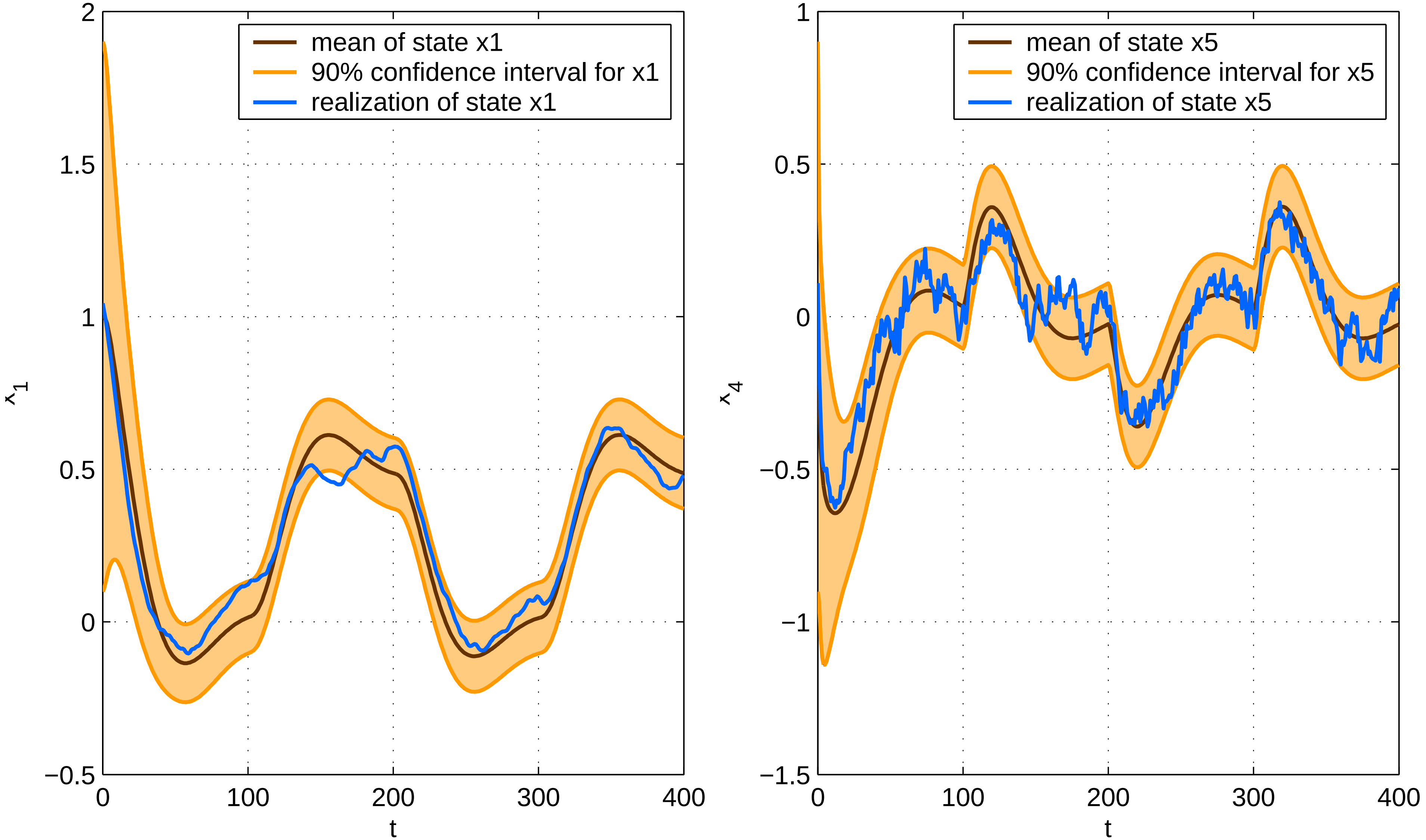
at each time t plot

- actual state in this particular run $x(t)$
- mean state $\mu_{x_i}(t)$
- 90% confidence interval $[\mu_{x_i}(t) - h(t), \mu_{x_i}(t) + h(t)]$, where $h(t)$ is as usual

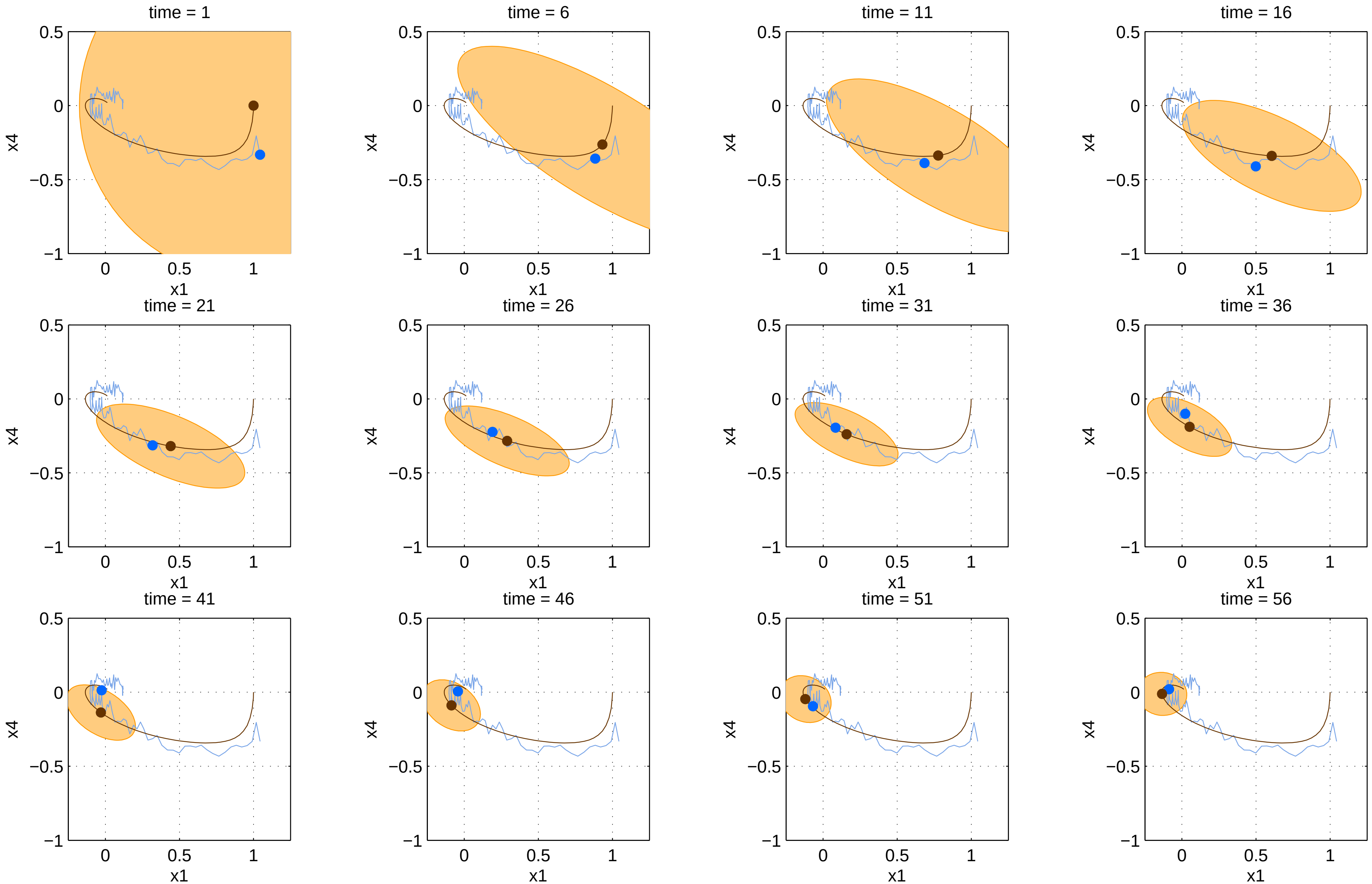
$$h(t) = \Big(\big(\Sigma_x(t)\big)_{ii} F^{-1}_{\chi^2_1}(0.9)\Big)^{\frac{1}{2}}$$

Example: Stochastic Simulation of Mass-Spring System

position and velocity of mass 1



Example: Ellipsoids



Steady-State Behavior

the Lyapunov equation is the same as the one we used for *controllability* analysis

if A is stable, then the limit is

$$\Sigma_{xss} = \lim_{t\to\infty} \Sigma_x(t) = \sum_{k=0}^{\infty} A^k \Sigma_v (A^k)^T$$

the *steady-state* covariance

as in controllability, this is the unique solution to the Lyapunov equation

$$\boxed{\Sigma_{xss} - A\Sigma_{xss}A^T = \Sigma_v}$$

if $\Sigma_v = BB^T$ then Σ_{xss} is the *controllability Gramian*

Stochastic processes

A *stochastic process* is an *infinitely long random vector*.

It has mean and covariance

$$\mathbf{E}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \mu_x(0) \\ \mu_x(1) \\ \mu_x(2) \\ \vdots \end{bmatrix} \qquad \mathbf{cov}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \Sigma_x(0,0) & \Sigma_x(0,1) & \dots \\ \Sigma_x(1,0) & \Sigma_x(1,1) & \\ \Sigma_x(2,0) & \Sigma_x(2,1) & \\ \vdots & & \end{bmatrix}$$

- For each $w \in \Omega$, the random variable x returns of the *entire sequence* $x(0), x(1), \dots$
- If $x(0), x(1), \dots$ are Gaussian and IID, then x is called *white Gaussian noise* (WGN)

  In this case, $\Sigma_x(i,j) = 0$ if $i \neq j$

Generating stochastic processes

Suppose $v(0), v(1), \dots$ is white Gaussian noise, with covariance $\mathbf{cov}(v(t)) = I$, and

$$\begin{aligned} x(t+1) &= Ax(t) + Bv(t) \qquad\qquad x(0) = 0 \\ y(t) &= Cx(t) \end{aligned}$$

We have $y = Tv$, where T is the Toeplitz matrix

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & & & & \\ H(1) & 0 & & & \\ H(2) & H(1) & 0 & & \\ H(3) & H(2) & H(1) & 0 & \\ \vdots & & & & \ddots \end{bmatrix} \begin{bmatrix} v(0) \\ v(1) \\ v(2) \\ v(3) \\ \vdots \end{bmatrix}$$

and $H(0), H(1), \dots$ is the *impulse response*

$$H(t) = \begin{cases} CA^{t-1}B & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

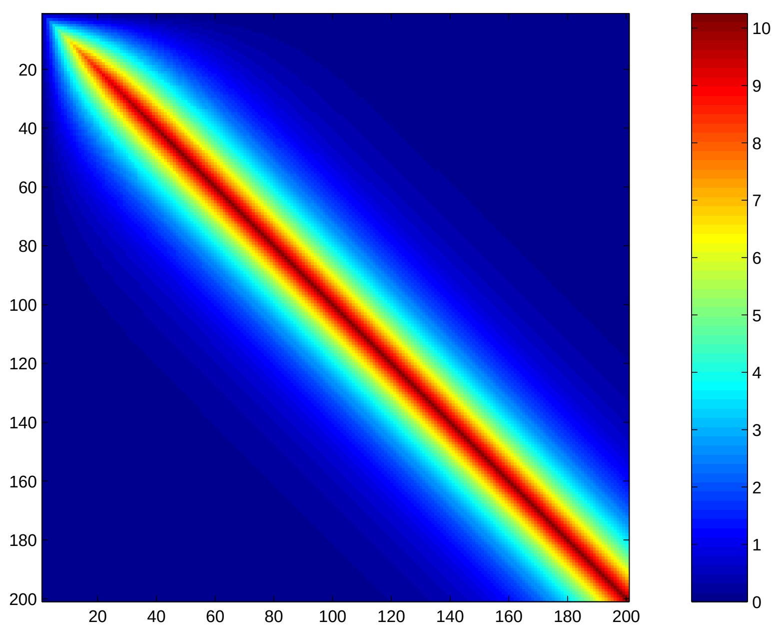
Example

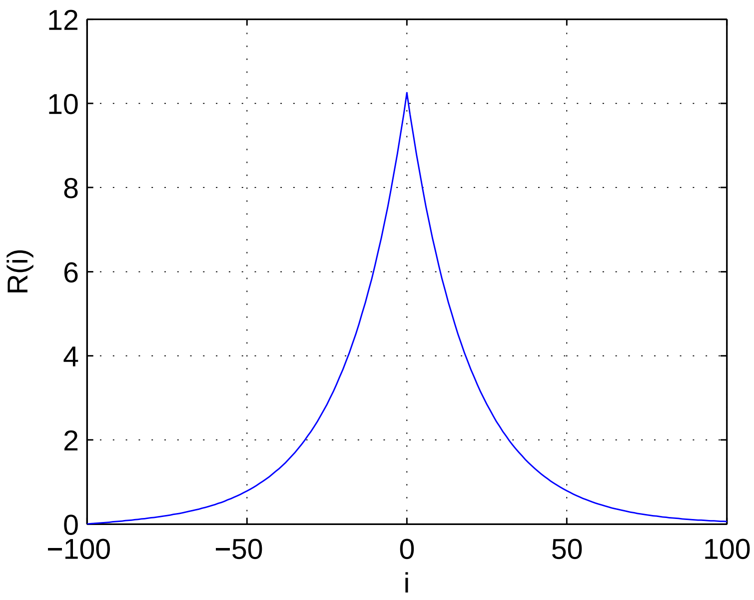
Let $W = \mathbf{cov}(y) = TT^T$. For example, suppose

$$x(t+1) = 0.95x(t) + v(t) \qquad x(0) = 0$$
$$y(t) = x(t)$$

An image plot of $W(i,j)$, is below. Notice that it becomes constant along diagonals.

Also plotted is $R(i) = W(i+100, i)$





The Output Covariance

We have the covariance of the output $W = \mathbf{cov}(y)$ satisfies

$$W(i,j) = \sum_{k=0}^{\infty} T_{ik}(T_{jk})^T$$

$$= \sum_{k=0}^{\infty} H(i-k)\big(H(j-k)\big)^T$$

since $T(i,j) = H(i-j)$. Therefore, evaluating W along the j'th diagonal

$$W(i+j,i) = \sum_{k=0}^{\infty} H(i+j-k)\big(H(i-k)\big)^T$$

$$= \sum_{p=-i}^{\infty} H(j-p)H^T(-p)$$

and so

$$\lim_{i\to\infty} W(i+j,i) = \sum_{p=-\infty}^{\infty} H(j-p)H^T(-p)$$

The Output Covariance

Let

$$R(j) = \sum_{p=-\infty}^{\infty} H(j-p)H^T(-p)$$

Then we have

$$\lim_{i\to\infty} \mathbf{cov} \begin{bmatrix} y(i) \\ y(i+1) \\ y(i+2) \\ y(i+3) \\ \vdots \end{bmatrix} = \begin{bmatrix} R(0) & R(-1) & R(-2) & \dots & \\ R(1) & R(0) & R(-1) & R(-2) & \dots \\ R(2) & R(1) & R(0) & R(-1) & \\ R(3) & R(2) & R(1) & R(0) & \\ \vdots & & & & \ddots \end{bmatrix}$$

- We must have $R(i) = R(-i)^T$
- As i becomes large, the output covariance W becomes Toeplitz

Asymptotic Stationarity

This means that the pdf of

$$\begin{bmatrix} y(i) \\ y(i+1) \\ y(i+2) \\ \vdots \\ y(i+N) \end{bmatrix}$$

tends to a limit as i becomes large.

The output process is therefore called *asymptotically stationary*.

Stationary Stochastic Processes

A stochastic process $y(0), y(1), \ldots$ is called *stationary* if for every i and every $N > 0$ the pdf of

$$\begin{bmatrix} y(i) \\ y(i+1) \\ y(i+2) \\ \vdots \\ y(i+N) \end{bmatrix}$$

is independent of i.

- If $y(0), y(1), \ldots$ is stationary, then

  $$W(i,j) = \mathbf{cov}\big(y(i), y(j)\big) = R(i-j)$$

  for some sequence of matrices $R(0), R(1), \ldots$

- Any segment of the signal $y(i), \ldots, y(i+N)$ has the same statistical properties as any other

- R is called the *autocovariance* of the process

State-Space Formulae

Suppose $v(0), v(1), \ldots$ is white Gaussian noise, with covariance $\mathbf{cov}(v(t)) = I$, and

$$\begin{aligned} x(t+1) &= Ax(t) + v(t) \qquad\qquad x(0) \sim \mathcal{N}(0, \Sigma_x(0)) \\ y(t) &= Cx(t) \end{aligned}$$

We have

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & & & & \\ I & 0 & & & \\ A & I & 0 & & \\ A^2 & A & I & 0 & \\ \vdots & & & & \ddots \end{bmatrix} \begin{bmatrix} v(0) \\ v(1) \\ v(2) \\ v(3) \\ \vdots \end{bmatrix} + \begin{bmatrix} I \\ A \\ A^2 \\ A^3 \\ \vdots \end{bmatrix} x(0)$$

$$= P \begin{bmatrix} v(0) \\ v(1) \\ \vdots \end{bmatrix} + Jx(0)$$

State-Space Formulae

Therefore $x(0), x(1), x(2), \ldots,$ is a stochastic process with covariance

$$\mathbf{cov}\begin{bmatrix} x(0) \\ x(1) \\ \vdots \end{bmatrix} = P \begin{bmatrix} \Sigma_v & & & \\ & \Sigma_v & & \\ & & \ddots & \\ & & & \Sigma_v \end{bmatrix} P^T \; + \; J\Sigma_x(0)J^T$$

Hence for $i \geq j$

$$\begin{aligned} \mathbf{cov}\big(x(i), x(j)\big) &= \sum_{k=1}^{j} A^{i-1-k}\Sigma_v(A^{j-1-k})^T + A^{i-1}\Sigma_x(0)(A^{j-1})^T \\ &= A^{i-j}\left(\sum_{k=1}^{j} A^{j-1-k}\Sigma_v(A^{j-1-k})^T + A^{j-1}\Sigma_x(0)(A^{j-1})^T\right) \\ &= A^{i-j}\Sigma_x(j) \end{aligned}$$

similarly, for $i \leq j$ we have $\mathbf{cov}\big(x(i), x(j)\big) = \Sigma_x(j)(A^{j-i})^T$.

State Covariance

Hence we have the state covariance

$$\mathbf{cov}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \Sigma_x(0) & \Sigma_x(1)A^T & \Sigma_x(2)(A^2)^T & \Sigma_x(3)(A^3)^T \\ A\Sigma_x(0) & \Sigma_x(1) & \Sigma_x(2)A^T & \Sigma_x(3)(A^2)^T \\ A^2\Sigma_x(0) & A\Sigma_x(1) & \Sigma_x(2) & \Sigma_x(3)A^T \\ \vdots & \ddots & \ddots & \end{bmatrix}$$

As $t \to \infty$ we have $\Sigma_x(t) \to \Sigma_{xss}$, so

$$\lim_{t\to\infty} \mathbf{cov}\begin{bmatrix} x(t) \\ x(t+1) \\ x(t+2) \\ x(t+3) \\ \vdots \end{bmatrix} = \begin{bmatrix} \Sigma_{xss} & \Sigma_{xss}A^T & \Sigma_{xss}A^{T^2} & \Sigma_{xss}A^{T^3} \\ A\Sigma_{xss} & \Sigma_{xss} & \Sigma_{xss}A^T & \Sigma_{xss}A^{T^2} \\ A^2\Sigma_{xss} & A\Sigma_{xss} & \Sigma_{xss} & \Sigma_{xss}A^T \\ A^3\Sigma_{xss} & A^2\Sigma_{xss} & A\Sigma_{xss} & \Sigma_{xss} \\ \vdots & \ddots & \ddots & \end{bmatrix}$$

The Autocovariance

The autocovariance of the output y is

$$R(i) = CA^i \Sigma_{xss} C^T \qquad \text{for } i \geq 0$$

where Σ_{xss} is the unique solution to the Lyapunov equation

$$\Sigma_{xss} - A \Sigma_{xss} A^T = I$$

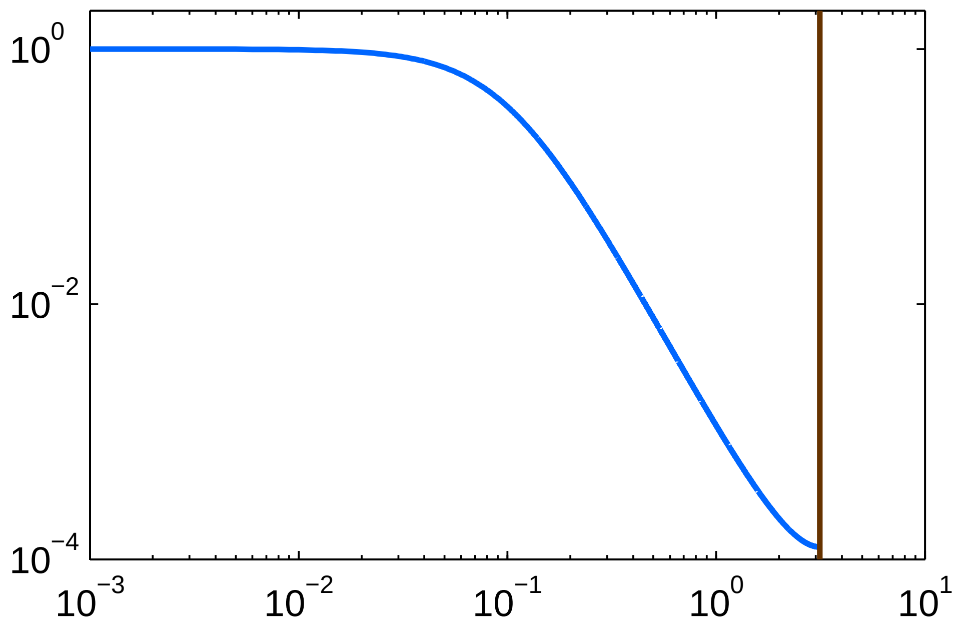
Example: Low-Pass Filter

Let's look at the low-pass filter

$$\hat{G}(z) = \frac{c}{(z - e^{-\lambda})^3}$$

where $\lambda = 0.1$.

- The breakpoint frequency is 20π rad s^{-1} when sampling period $h = 1$.
- The constant c is chosen such that $\hat{G}(1) = 1$.



Example: Low-Pass Filter

The input, output, autocorrelation, and breakpoint frequency are below.